

Summer Packet

AP Calculus BC

Summer 2022

Enclosed is a packet that needs to be completed this summer. You may use any resources that are available to you. Some questions may require the use of a calculator. I will collect it the first day of class in September, and I will grade it. This will be your first Calculus grade.

Enjoy your summer. See you in the Fall.

Mr. A

Sequences

Example #1:

List the 1st five terms of the following sequences:

a) $\{a_n\} = \{3 + (-1)^n\}$

b) $\{a_n\} = \left\{ \frac{2n}{1+n} \right\}$

c) $\{a_n\} = \left\{ \frac{n^2}{2^n - 1} \right\}$

Example #2

Find the limit of the sequence whose n th term is: $a_n = \left(1 + \frac{1}{n}\right)^n$

Example #3

Determine whether the following sequences converge or diverge:

a) $\{a_n\} = \{3 + (-1)^n\}$

b) $\{a_n\} = \left\{ \frac{n}{1-2n} \right\}$

Example #4

Show that the sequence whose n th term is $\{a_n\} = \left\{ \frac{n^2}{2^n - 1} \right\}$ converges.

Example #5

Recall:

Does the sequence $\{a_n\} = \left\{ \frac{(-1)^n}{n!} \right\}$ converge?

Example #6

Find a sequence whose first 5 terms are $\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}$ Does it converge?

Example #7

Determine the n th term of the sequence whose first five terms are:

$-\frac{2}{1}, -\frac{8}{6}, -\frac{26}{24}, -\frac{80}{120}, -\frac{242}{120}, \dots$ Then decide whether it converges or diverges.

Monotonic Sequences and Bounded Sequences

Definition of a Monotonic Sequence:

Example #8

Determine whether each sequence with the given n th term is monotonic:

$$\text{a) } \{a_n\} = \{3 + (-1)^n\} \quad \text{b) } \{a_n\} = \left\{ \frac{2n}{1+n} \right\} \quad \text{c) } \{a_n\} = \left\{ \frac{n^2}{2^n - 1} \right\}$$

Definition of a Bounded Sequence:

THEOREM:

Example #9

Discuss whether the following sequences are bounded and/or monotonic.

$$\text{a) } \{a_n\} = \left\{ \frac{1}{n} \right\} \quad \text{b) } \{a_n\} = \left\{ \frac{n^2}{(n+1)} \right\} \quad \text{c) } \{a_n\} = \{(-1)^n\}$$

Series and Convergence**Infinite Series:**

Definition of Convergent or Divergent Series

Example #1

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Example #2

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Example #3

$$\sum_{n=1}^{\infty} 1$$

Example #4

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Geometric Series:Convergence of a Geometric Series

Example #5

Test if the following geometric sequence converges $\sum_{n=0}^{\infty} \frac{3}{2^n}$. If it does, find its sum.

Example #6

Test if the following geometric sequence converges $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$. If it does, find its sum.

Example #7

Use a Geometric Series to write $.0\overline{80808}$ as a fraction.

***n*th-Term Test for Divergence**

Limit of *n*th Term of a Convergent Sequence

*n*th-Term Test for Divergence

Example #8

Examine the following series:

a)
$$\sum_{n=0}^{\infty} 2^n$$

b)
$$\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$$

c)
$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Example #9

A ball is dropped from a height of 6 feet and begins bouncing. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance travelled by the ball.

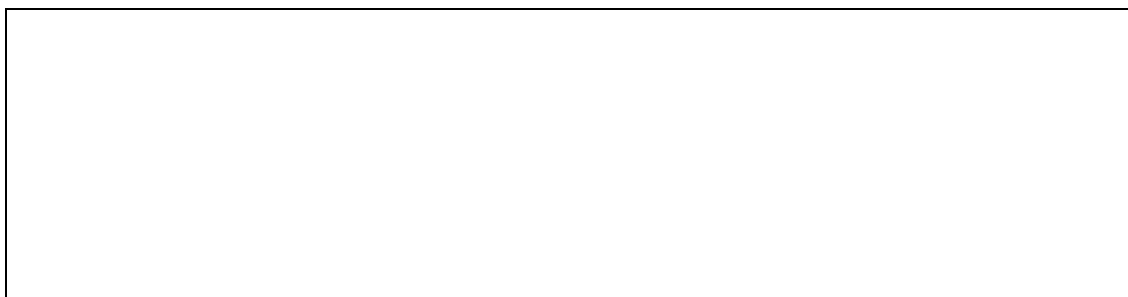
The Integral Test and p -series**The Integral Test**

Example #1

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$.

Example #2

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

***p*-Series and Harmonic Series****Example #3**

Do the following *p*-series converge or diverge?

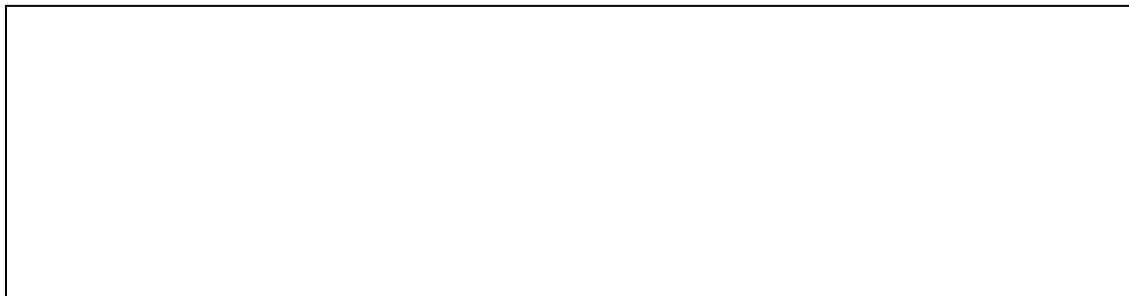
a.
$$\sum_{n=1}^{\infty} \frac{1}{n}$$

b.
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Example #4

Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

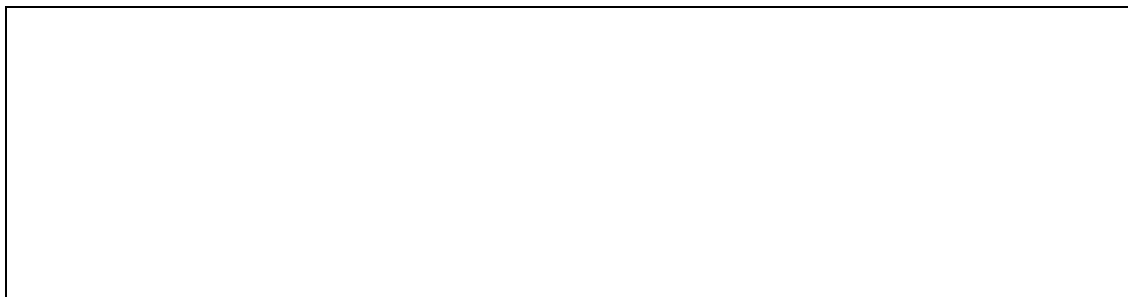
Comparisons of Series**Direct Comparison Test**

Example #1

Determine convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$.

Example #2

Determine convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$.

Limit Comparison Test

Example #3

Show that the following harmonic series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a > 0, \quad b > 0$$

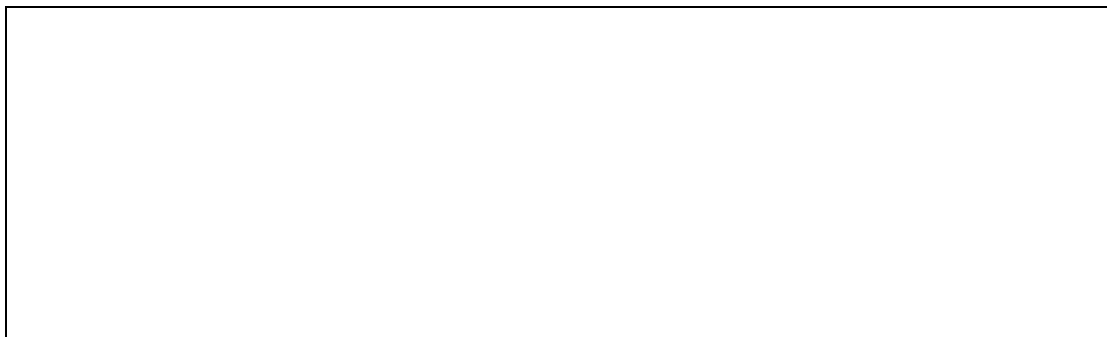
RECOGNITION:

Example #4

Determine convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$.

Example #5

Determine convergence or divergence of $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}$.

Alternating Series

Example #1

Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Example #2

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$.

Example #3

a) Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$.

Alternating Series Remainder

Example #4

Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$$

Absolute and Conditional Convergence**Example #5**

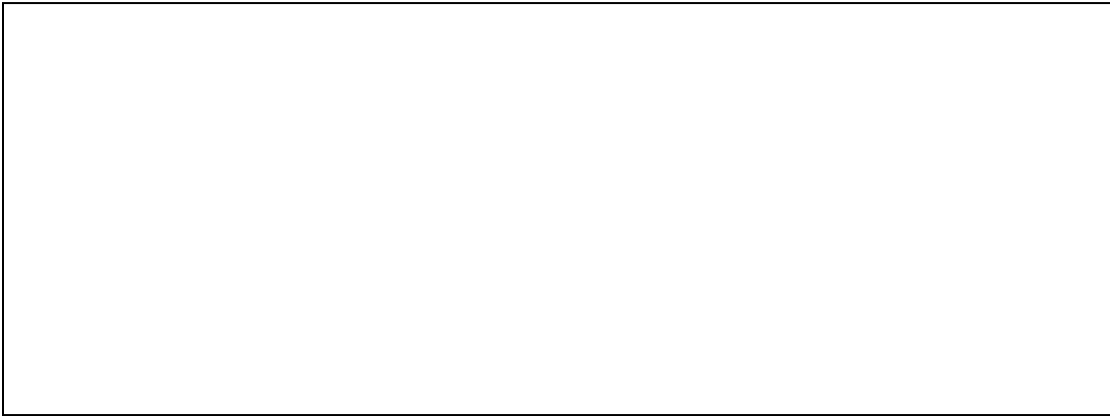
Determine whether each series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$$

b)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

c)
$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$$

d)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Ratio and Root Tests**The Ratio Test**

Example #1

Determine the convergence or divergence of $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Example #2

Determine whether each of the following series converge or diverge.

A) $\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

B) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Example #3

Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$.

The Root Test

Example #4

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the n th term approach 0? If not, series diverges.
2. Is the series one of the special types – geometric, p -series, telescoping, or alternating?
3. Can the Integral Test, the Root test or the Ratio test be applied?
4. Can the series be compared favorably to one of the special types?

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Example #5

Determine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ b. $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ c. $\sum_{n=1}^{\infty} ne^{-n^2}$ d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

e. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ f. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Taylor Polynomials and Approximations**Polynomial Approximations of Elementary Functions**

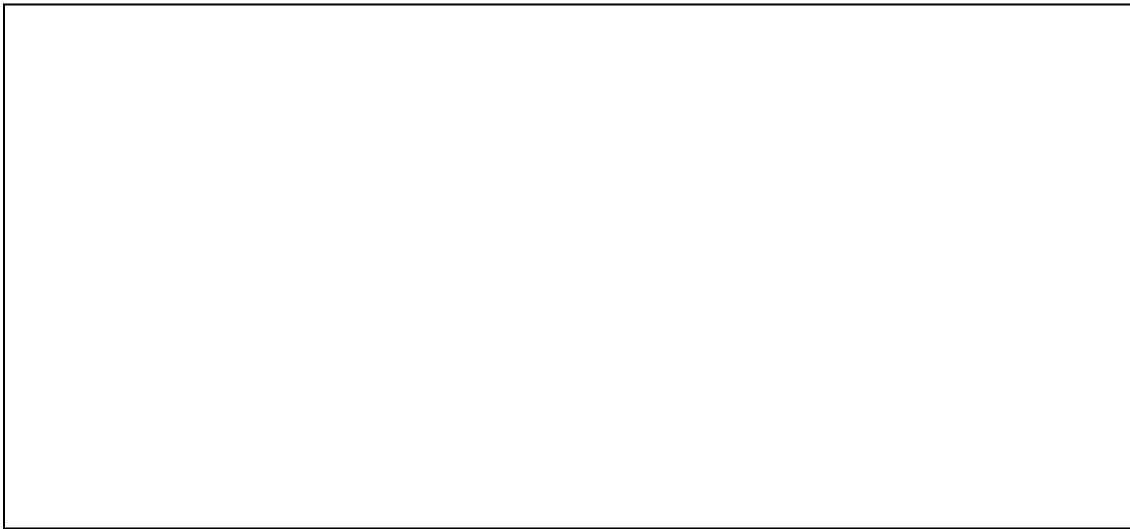
Example #1

For the function $f(x) = e^x$, find a first-degree polynomial function, $P_1(x) = a_0 + a_1x$, whose value and slope agree with the value of f at $x = 0$.

Example #2

Construct a table comparing the values of the polynomial

 $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$ with $f(x) = e^x$ for several values near 0.

Taylor and Maclaurin Polynomials

So the n th Maclaurin polynomial for $f(x) = e^x$ is given by:

Example #3

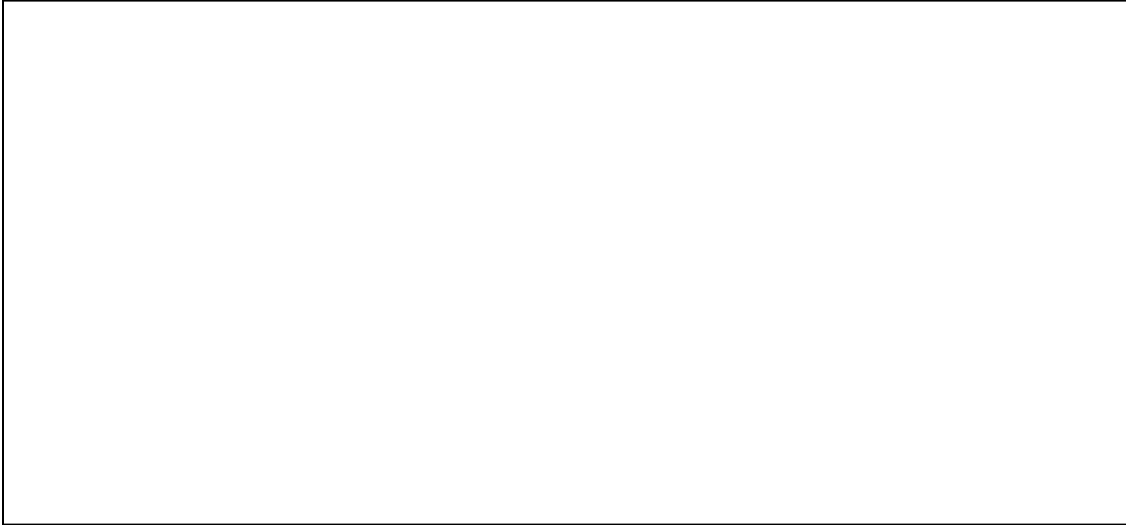
Find the Taylor polynomials $P_0, P_1, P_2, P_3,$ and P_4 for $f(x) = \ln x$ centered at $c = 1$.

Example #4

Find the Maclaurin Polynomials $P_0, P_2, P_4,$ and P_6 for $f(x) = \cos x$. Use P_6 to approximate the value of $\cos(0.1)$.

Example #5

Find the 3rd Taylor polynomial for $f(x) = \sin x$ expanded about $c = \frac{\pi}{6}$.

Remainder of a Taylor Polynomial**Example #6**

The third Maclaurin polynomial for $\sin x$ is given by $P_3(x) = x - \frac{x^3}{3!}$. Use Taylor's

Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the accuracy of the approximation.

Example #7

Determine the degree of the Taylor polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the Lagrange Error is less than 0.001.

Power Series

Examples of Power Series:

a.

b.

c.

Radius and Interval of Convergence



Examples:

1) Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

2) Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$.

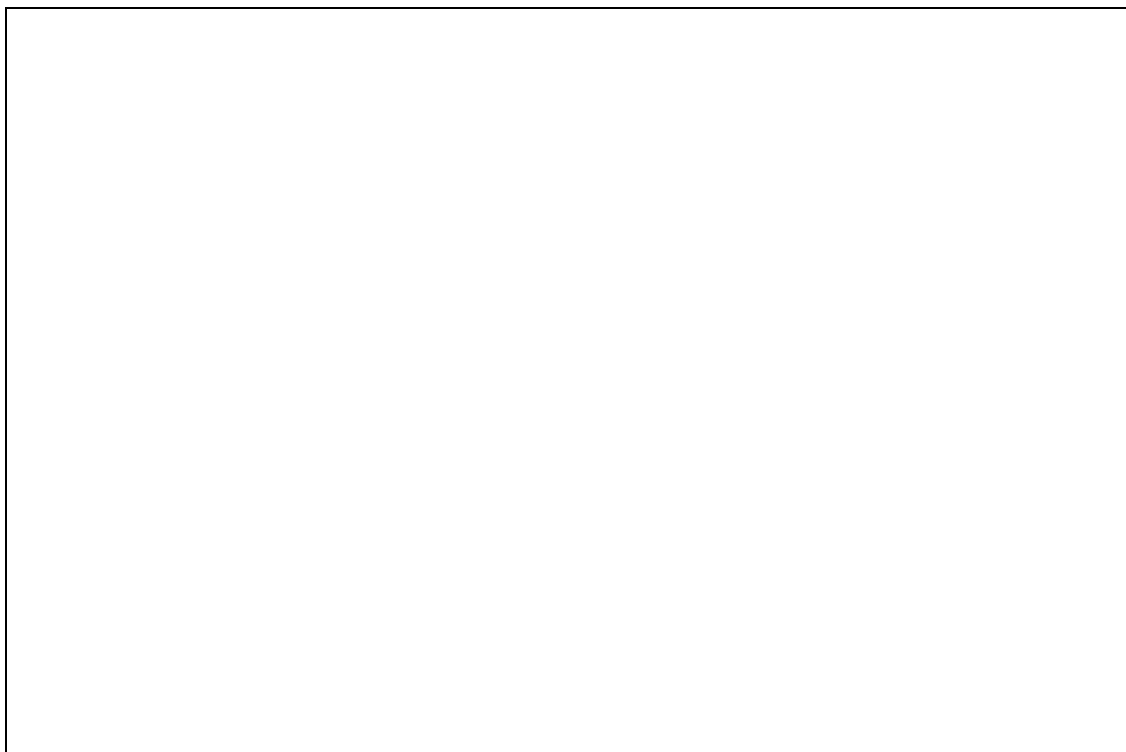
3) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Endpoint Convergence

Examples:

4) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.5) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$.6) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

Intervals of Convergence for $f(x)$, $f'(x)$ and $\int f(x)$.



Example

Given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Find the Intervals of convergence of $f(x)$, $f'(x)$ and $\int f(x)$.

Representation of Functions by Power Series

Recall:

Examples:

1) Find a power series for $f(x) = \frac{4}{x+2}$, centered at 0.

2) Find a power series for $f(x) = \frac{1}{x}$, centered at 1.

3) Find a power series, centered at 0, for $f(x) = \frac{3x-1}{x^2-1}$.

4) Find a power series for $f(x) = \ln x$, centered at 1.

5) Find a power series for $g(x) = \arctan x$, centered at 0.

6) Find a power series for $f(x) = \frac{4x-7}{2x^2+3x-2}$, centered at 0.

7) Find a power series for $f(x) = \frac{4}{4+x^2}$, centered at 0.

8) Find a power series for $f(x) = \frac{1}{2x-5}$, centered at 0.

9) Find a power series for $f(x) = \frac{3}{4-x}$, centered at $c = -2$.

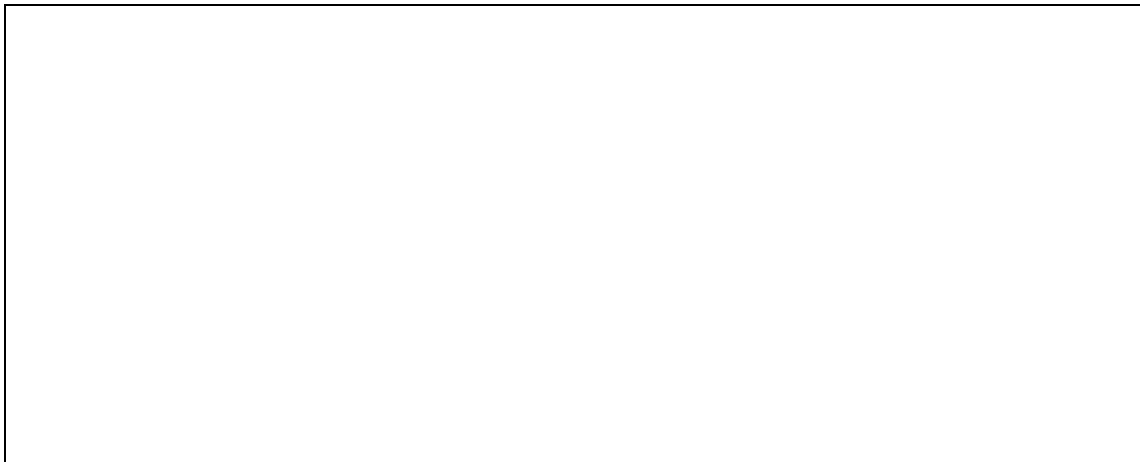
Taylor and Maclaurin Series

Example #1

Use the function $f(x) = \sin x$ to form the Maclaurin series

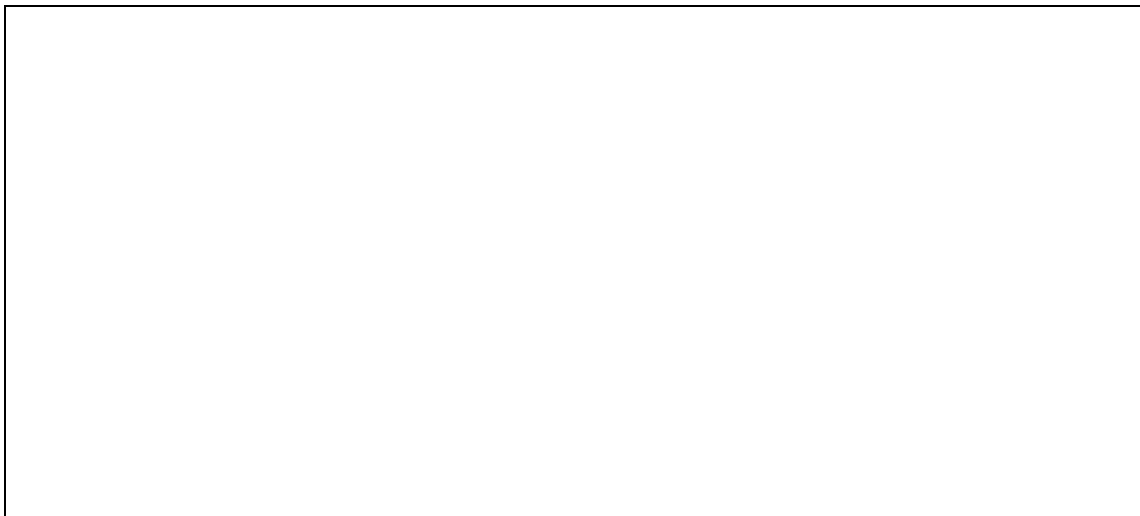
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

and determine the interval of convergence.



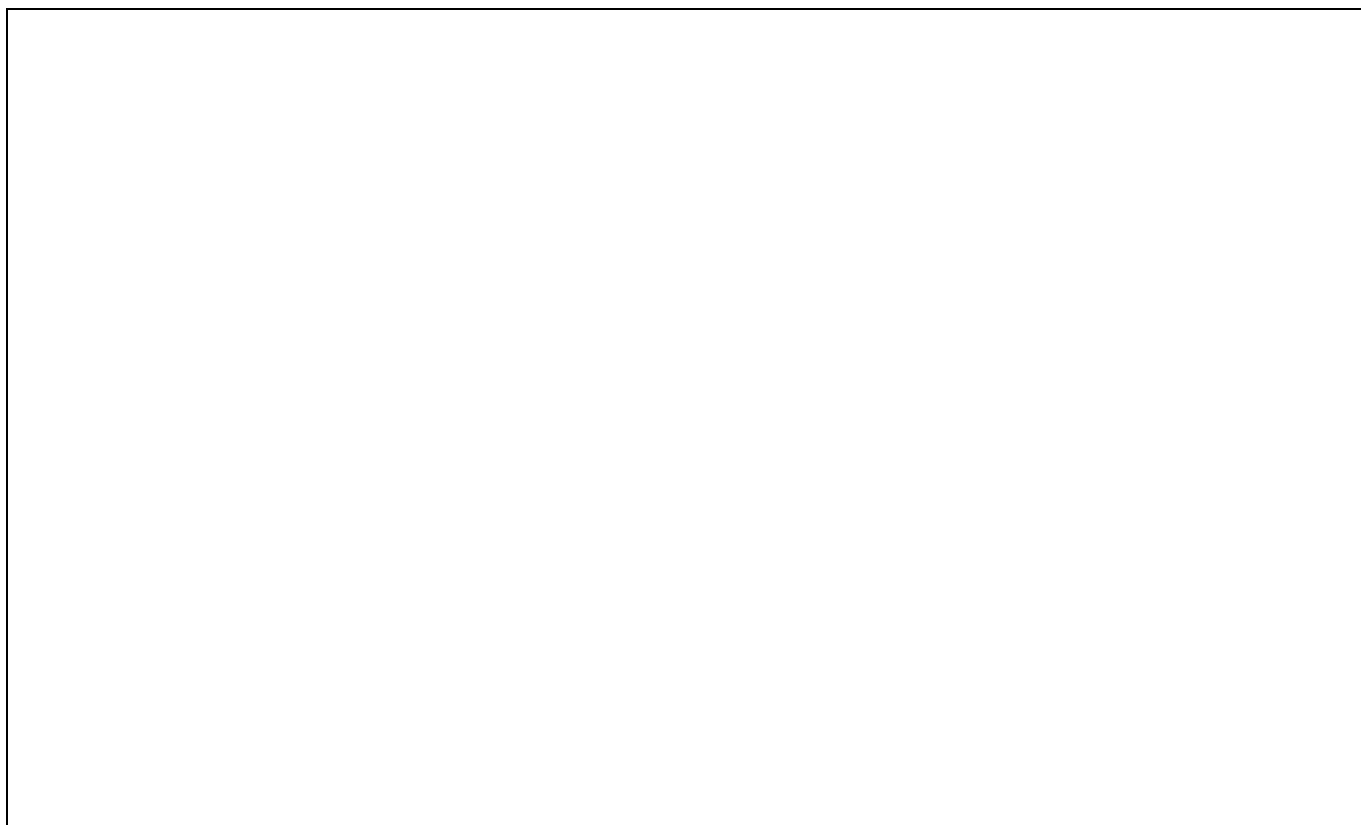
Example #2

Show that the Maclaurin series $f(x) = \sin x$ converges to $\sin x$ for all x .



Example #3

Find the Maclaurin series for $f(x) = \sin x^2$.



Example #4

Find the power series for $f(x) = \cos\sqrt{x}$.

Example #5

Find the power series for $f(x) = \sin^2 x$.

Example #6

Use a power series to approximate $\int_0^1 e^{-x^2} dx$ with an error of less than 0.01.

Class Work:

Find the power series for $f(x) = \ln x$, centered at $c = 1$.

Find the Maclaurin series for $f(x) = e^{2x}$.

Find the power series for $f(x) = e^x$, centered at $c = 1$.